An Extension of Ausubel’s Auction for Heterogeneous Discrete Goods

Hakan İnal∗†
Department of International Trade and Finance
İzmir University
hakaneconomics@gmail.com
October 28, 2014

Abstract

Ausubel’s dynamic private-values auction for heterogeneous discrete goods, Ausubel (2006), yields an efficient equilibrium outcome but it is designed for a limited class of environments. If bidders’ values for bundles of goods are not integers, then Ausubel’s auction may end without allocating goods if no information on bidders’ values is available. In this paper, I extend Ausubel’s auction for heterogeneous discrete goods to real-valued quasilinear utility functions. The mechanism I propose reaches a Walrasian equilibrium price vector in finite “steps” without any additional information on bidders’ values. In the extension of Ausubel’s auction, truthful bidding constitutes an efficient equilibrium.

JEL Category D44

∗I would like to thank Ket Richter and Jan Werner for their continuous support, patience, and effort for my education. Their guidance has been enlightening. I would like to thank late Leo Hurwicz, and Andy McLennan for their contribution to my education. I benefited a lot from discussions with David Rahman and Itai Sher. I also would like to thank Pat Bajari, Tuba İnal, Ichiro Obara, Raj Singh, and the seminar participants at Central Michigan University, METU, the University of Minnesota, the Virginia Commonwealth University, Whitman College, the Games 2008, and the Society for Economic Design 2008 Conference for their valuable comments. All errors are mine.
†http://hakaninal.weebly.com
1 Introduction

Auctioning of multiple goods has become a rapidly developing part of the auction theory since the Federal Communications Commission began auctioning wireless communication bands in 1994. It is well-known that when many units of a good are to be auctioned, standard auction mechanisms are generally inefficient, i.e., they do not award goods to buyers who value them most. This inefficiency arises from buyers’ tendency to shade their bids, the demand-reduction problem (see Krishna (2002), Ausubel and Cramton (2002), and Milgrom (2004)). Vickrey’s sealed-bid auction, Vickrey (1961), is among the few exceptions immune to the demand-reduction problem. Even though the Vickrey auction is efficient, it is not widely used in practice. In the Vickrey auction, bidders are supposed to submit their whole demand curves to the auctioneer. So, if there are 10 objects for sale, each bidder must compute and submit his values for $2^{10}$ bundles of goods. Bidders prefer not to submit so much information about themselves. Ausubel (2006) introduced an elegant dynamic auction$^1$ for divisible and discrete heterogeneous goods, and it is immune to the demand-reduction problem. As Ausubel’s auction does not require bidders to reveal their whole demand correspondences, it preserves privacy of bidders, and it is simpler than the Vickrey auction. Also, bidders are allowed to exercise their market power but strategic bidders behave as price-takers under Ausubel’s nonlinear pricing rule, which solves the demand-reduction problem. However, Ausubel’s auction for heterogeneous discrete goods (hereafter, Ausubel’s auction) is designed (see Section VII, Ausubel (2006)) for a limited class of environments: Bidders’ values for bundles are restricted to be integers. The reason for this integer restriction is not explained in his paper. The tâtonnement algorithm$^2$ (price adjustment procedure) uses integer property of utility functions: At integer price vectors, the auctioneer asks bidders to submit their demand sets at the current price vector, and using these demand reports she determines which prices to change. If bidders’ values for bundles are not integers, without

---

$^1$See the example in Ausubel (2006), pp. 606-607 for an illustration of the auction.

information on these values, the tâtonnement algorithm may not reach a Walrasian equilibrium price vector. If a Walrasian equilibrium price vector is not reached, then the auction terminates without allocating goods because of the way it is designed. In order to overcome this problem, Ausubel’s auction may be generalized such that the auctioneer asks for demand reports at \( \epsilon \) price increments, and determines which prices to change. In discrete-time price adjustment, she changes prices in \( \epsilon \) increments and asks for demand reports at each increment to determine which prices to change by \( \epsilon \). In continuous-time price adjustment, she changes prices continuously, stops the clock at \( \epsilon \) price increments and asks for demand reports to determine which prices to adjust. Whether discrete- or continuous-time price adjustment procedure is used, this straightforward generalization of Ausubel’s auction still may not reach a Walrasian equilibrium, and consequently the auction may terminate without allocating goods. In Section 1.1, I provide an example illustrating these problems. In this example, for any rational price increment \( \epsilon \) the price adjustment procedure never reaches an equilibrium price vector, and Ausubel’s auction always terminates without allocating goods.

Alternatively, one may consider a continuous-time generalization of Ausubel’s auction where bidders report their demands continuously, and the auctioneer changes prices according to these reports continuously. Ausubel (2006), p.628 and Ausubel (2005), p.12 show that in Ausubel’s auction, a Lyapunov function whose minimizers coincide with Walrasian equilibrium price vectors decreases by a positive integer amount at each integer price vector reached. Therefore, the price adjustment procedure reaches a Walrasian equilibrium in finite time. But when values and prices are real, this argument is no longer valid for the continuous-time generalization of Ausubel’s auction, and it is not necessarily “obvious” that the price adjustment procedure will reach a Walrasian equilibrium in finite time.

In this paper, I extend Ausubel’s auction to real-valued quasilinear utility functions by introducing a continuous-time analogous extension of the tâtonnement algorithm in Ausubel (2006), p.619. The main difference between the proposed continuous-time algorithm and the continuous-time algorithm in Ausubel (2006), p.620 is the timing of demand reports. In Ausubel’s version, which works with integer values, demand reports are collected and the set of goods whose prices will be changed are determined at each integer price vector reached. On the other hand, in the proposed algorithm bidders report their demands whenever they add a new bundle of goods to their demand set during the price adjustment, and the auctioneer recalculates at that
moment what prices to change. I show that the extended tâtonnement algorithm reaches a Walrasian equilibrium price vector in finite “steps” when bidders’ values are real numbers. The extended Ausubel auction for heterogeneous discrete goods has an efficient equilibrium and yields Walrasian equilibrium prices when bidders’ values for bundles are real numbers. Unlike the tâtonnement algorithm of Ausubel (2006), pp. 619-620, which uses integer property of bidders’ values, in the extended tâtonnement algorithm the auctioneer does not need any information on bidders’ values. My paper draws heavily from Ausubel (2006), but the extension builds on Ausubel’s auction to improve it. Results here are parallel to the results in Ausubel (2006): some of them are simple modifications of those in Ausubel (2006), and they are clearly stated. Some of the results are not straightforward, including the main result, Theorem 1, and their proofs are given.

In Ausubel’s auction, Ausubel (2006), bidders submit their demands as prices are adjusted. A bidder is credited a unit of a good at the current price when the rest of the bidders lower their demand for this good. A unit of a good is debited from a bidder at the current price when the rest of the bidders increase their demand for this good. The auctioneer calculates the set of goods in excess demand and adjusts the prices accordingly. The auction ends whenever there is a market clearing allocation demanded by bidders at the current price. Bidders are assumed to have private values for goods (each bidder’s values for goods depend only on his own type) and have utility functions quasilinear in money. In the case of divisible goods, bidders have concave utility functions whereas in the case of discrete goods, they have preferences satisfying the gross substitutes condition. The gross substitutes assumption basically requires that a bidder’s demand for a set of goods to be nondecreasing if their prices remain the same while the rest of the prices do not decrease. This assumption guarantees the existence of a Walrasian equilibrium (see Gul and Stacchetti (1999)). When goods are divisible, the classical Walrasian tâtonnement is used to determine the path of prices. In the case of discrete goods, a tâtonnement algorithm (see Ausubel (2005)) is used. There are two analogous versions of the tâtonnement algorithm: the ascending algorithm and the descending algorithm. Ausubel (2005) shows that the ascending (descending) algorithm reaches a Walrasian equilibrium price vector in finite steps if the initial prices are sufficiently small (large). Ausubel (2006) converts this discrete-time price adjustment procedure to a continuous-time price adjustment procedure. He achieves that by linearly increasing (or decreasing) prices between two consecutive integer-
valued price vectors determined by the tâtonnement algorithm. Then, he uses this continuous-time price adjustment procedure to prove that sincere bidding by bidders comprises an efficient equilibrium and yields Walrasian equilibrium prices.

In the proposed extension of Ausubel’s auction, the auction starts at an initial price vector \( p(0) \). Each bidder submits his report, a set of bundle of goods he demands at \( p(0) \). The auctioneer adjusts prices continuously according to the extended tâtonnement algorithm using these demand reports. Bidders may add a new bundle to or remove a bundle from their demand set during the price adjustment, and they are required to submit their reports whenever they add a new bundle to their demand sets as prices change. At any time \( t \in [0, \infty) \), if there is a bidder who submits a new report at the current price \( p(t) \), then the auctioneer stops the price adjustment. The auctioneer receives demand reports from bidders, and prices are adjusted continuously according to the extended tâtonnement algorithm. During the price adjustment process, for each bidder \( i \in N \), if opponents of bidder \( i \) lower their demand for a good, then the good is credited to bidder \( i \) at price \( p(t) \). On the other hand, if opponents of bidder \( i \) rise their demand for a good, then it is debited from bidder \( i \) at price \( p(t) \). The auction ends at time \( T \in [0, \infty) \) when there is a market clearing allocation of goods in these reports made at time \( T \). Payment of each bidder is calculated by adding his credits and debits, and market clearing allocation of goods are made.

Section 1.1 shows how an \( \epsilon \) price-increment generalization of Ausubel’s auction may not work if a specific information on bidders’ values is not available to the auctioneer. Section 2 gives the assumptions of the model. In Section 3, the extension of Ausubel’s auction and the extended tâtonnement algorithm are explained. The dynamic game and the equilibrium concept are also defined. Section 4 explains how the extended ascending and descending algorithms determine price paths, and shows that they reach Walrasian equilibrium price vectors in finite steps. In Section 4.3, the tâtonnement algorithm in Ausubel’s auction and the extended tâtonnement algorithm proposed in this paper are compared, and discussion as to why truthful bidding constitutes an efficient equilibrium is given.

### 1.1 An Example

When Ausubel’s auction is generalized to \( \epsilon \) price-increments, the auctioneer asks for demand reports at \( \epsilon \) price-increments. Using these demand re-
ports, she determines the smallest set of prices whose $\epsilon$ increase decrease the Lyapunov function (see Ausubel (2006), p.618) most. The example in this section shows that this $\epsilon$ price-increment generalization of Ausubel’s auction may never reach a Walrasian equilibrium price vector for any choice of rational-valued $\epsilon$. Moreover, by the design of Ausubel’s auction, goods are never allocated when the price adjustment stops. The result does not change whether prices are increased continuously or in $\epsilon$ increments as the auctioneer asks for demand reports at $\epsilon$ price-increments.

There are two types of goods, and two bidders with identical quasilinear utility functions. A unit supply of each good is available. Bidders’ utilities are $U(1, 1) = \sqrt{5}$, $U(1, 0) = \sqrt{3}$, $U(0, 1) = \sqrt{2}$, and $U(0, 0) = 0$. Observe that the efficient allocations are $((1, 0), (0, 1))$, and $((0, 1), (1, 0))$. Suppose that the auctioneer chooses the size $\epsilon > 0$ of the grid of prices such that there does not exist an integer $K$ such that $\epsilon K = \sqrt{3} - \sqrt{2}$. Since the auctioneer does not know bidders’ values, she typically does not choose $\epsilon$ violating this condition. Note that all rational $\epsilon$ satisfy this restriction. The set of Walrasian equilibrium prices are

$$\{(p_1, p_2) \in \mathbb{R}^2 | p_2 = p_1 - (\sqrt{3} - \sqrt{2}), \text{ and } p_1 \in [\sqrt{5} - \sqrt{2}, \sqrt{3}], \text{ and } p_2 \in [\sqrt{5} - \sqrt{3}, \sqrt{2}]\}.$$ 

Hence, the lowest Walrasian equilibrium price vector is

$$p = (\sqrt{5} - \sqrt{2}, \sqrt{5} - \sqrt{3}).$$

Define the following sets of prices:

$$P_{11} = \{(p_1, p_2) \in \mathbb{R}^2 | 0 \leq p_2 \leq \sqrt{5} - \sqrt{3}, \text{ and } 0 \leq p_1 \leq \sqrt{5} - \sqrt{2}\},$$

$$P_{10} = \{(p_1, p_2) \in \mathbb{R}^2 | p_2 \geq \sqrt{5} - \sqrt{3}, \text{ and } 0 \leq p_1 \leq \sqrt{3}, \text{ and } p_2 \geq p_1 - (\sqrt{3} - \sqrt{2})\},$$

$$P_{01} = \{(p_1, p_2) \in \mathbb{R}^2 | p_1 \geq \sqrt{5} - \sqrt{2}, \text{ and } 0 \leq p_2 \leq \sqrt{2}, \text{ and } p_2 \leq p_1 - (\sqrt{3} - \sqrt{2})\},$$

$$P_{00} = \{(p_1, p_2) \in \mathbb{R}^2 | p_1 \geq \sqrt{3}, \text{ and } p_2 \geq \sqrt{2}\}.$$ 

Observe that if $p_t \in P_{ij}$ for some $i, j \in \{0, 1\}$, then $\{(i, j)\} \in Q(p_t)$, bidders’ demand set at price vector $p_t$. By Ausubel’s ascending algorithm (see Ausubel (2006), page 619), with a straightforward generalization to the grid size is $\epsilon > 0$, the next price vector reached after $p_t$ is

$$p_{t+1} = \begin{cases} 
  p_t + (\epsilon, \epsilon) & \text{if } p_t \in P_{11}, \\
  p_t + (\epsilon, 0) & \text{if } p_t \in P_{10}, \\
  p_t + (0, \epsilon) & \text{if } p_t \in P_{01}, \\
  p_t & \text{if } p_t \in P_{00}.
\end{cases}$$
For each initial price vector $p_0$ such that $p_0 \leq p$, Ausubel's ascending algorithm does not reach any Walrasian equilibrium price vector, and terminates at a price vector $p_T \in P_{00}$. Note that $Q(p_T) = \{(0,0)\}$. Since Ausubel's auction allocates goods according to the market clearing allocation at the equilibrium price vector reached, goods in this example cannot be allocated when the auction ends. In order to illustrate the problem, let $\epsilon = \frac{1}{7}$, and the initial price vector $p_0 = (0,0) \in P_{11}$. Then, Ausubel's ascending algorithm reaches $p_1 = (\frac{1}{4},\frac{1}{7}) \in P_{11}$, $p_2 = (\frac{2}{7},\frac{2}{7}) \in P_{11}$, $p_3 = (\frac{3}{4},\frac{3}{7}) \in P_{10}$, $p_4 = (1,\frac{2}{7}) \in P_{10}$, $p_5 = (1\frac{1}{4},\frac{3}{7}) \in P_{10}$, $p_6 = (1\frac{1}{4},1) \in P_{10}$, $p_7 = (1\frac{2}{7},1) \in P_{01}$, $p_8 = (1\frac{1}{4},1\frac{1}{7}) \in P_{10}$, $p_9 = (1\frac{1}{4},1\frac{1}{7}) \in P_{01}$, and terminates at $p_{10} = (1\frac{2}{7},1\frac{2}{7}) \in P_{00}$. Observe that Ausubel's ascending algorithm jumps over equilibrium price vectors, and terminates at a non-equilibrium price vector.

2 The Model

In this paper, I follow a notation very similar to the one in Ausubel (2005) so that the reader can easily follow one paper after the other. Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ stand for the sets of integer, rational and real numbers respectively. There are finite number of goods, $K = \{1,2,\ldots,K\}$. There is a seller with supply $S = (S^k)_{k \in K} \in \mathbb{Z}^K_+$ of discrete heterogeneous goods, and she wants to sell them to a finite group of bidders, $N = \{1,2,\ldots,N\}$. Consumption set of bidder $i$ is $X_i = \{x \in \mathbb{Z}^K : 0 \leq x^k \leq p_i^k \text{ for all } k \in K\}$ where $(p_i^k)_{k \in K} \in \mathbb{Z}^K$, and it is bounded below and above. $x_i = (x_i^k)_{k \in K} \in X_i$ is a bundle bidder $i$ consumes.

The following assumptions are made for each bidder $i$:

A.1 Private Values: Bidder $i$’s utility function $u_i : X_i \times \mathbb{R} \to \mathbb{R}$ is a function of bundle $x_i \in X_i$ and money $t_i \in \mathbb{R}$ he consumes, and it does not depend on any information about other bidders.

A.2 Quasilinearity: $u_i(\cdot)$ is assumed to be quasilinear in money, i.e., there exists $U_i : X_i \to \mathbb{R}$ such that for each $x_i \in X_i$ and each $t_i \in \mathbb{R}$,

\[ u_i(x_i, t_i) = U_i(x_i) + t_i \]

where $U_i(x_i)$ is $i$’s value for bundle $x_i$.

A.3 Strict Monotonicity: For all $(x_i', t_i')$ and $(x_i, t_i) \in X_i \times \mathbb{R}$ such that $(x_i', t_i') \succeq (x_i, t_i)$,

\[ u_i(x_i', t_i') > u_i(x_i, t_i). \]
Bidder $i$'s indirect utility function at price vector $p = (p^k)_{k \in K} \in \mathbb{R}^+_K$ is

$$V_i(p) = \max_{x_i \in X_i} \{U_i(x_i) - p \cdot x_i\},$$

and his demand correspondence (demand set) at price vector $p \in \mathbb{R}^+_K$ is

$$Q_i(p) = \arg \max_{x_i \in X_i} \{U_i(x_i) - p \cdot x_i\}.$$  

A Walrasian equilibrium is $(p^*, x^*)$ where $p^*$ is equilibrium price vector and $x^* = (x_i)_{i \in N}$ is equilibrium allocation, such that for each bidder $i \in N$ $x_i^* \in Q_i(p^*)$, and $\sum_{i \in N} x_i^* = S$. Existence of Walrasian equilibrium heavily relies on the gross substitutes assumption. Gul and Stacchetti (1999) shows that the gross substitutes assumption is sufficient and “almost necessary” for the existence of the Walrasian equilibrium. In the gross substitutes assumption below, each good is assumed to be available in unit supply. The auction mechanism proposed in this paper, like Ausubel’s auction Ausubel (2006), allows multiple units of each good without loss of generality (for more on this see Bikhchandani and Mamer (1997), and the discussion on Ausubel (2006), p. 617). For more on the gross substitutes assumption see Milgrom and Strulovici (2009): they make an extensive analysis of the gross substitutes assumption and its relation to equilibrium.

A.4 Gross Substitutes: For all price vectors $p$ and $p' \in \mathbb{R}^+_K$ such that $p' \geq p$, if demand $Q_i(\cdot)$ is single-valued both at $p$ and at $p'$, $x_i \in Q_i(p)$, and $x'_i \in Q_i(p')$, then $x'_i k \geq x_i k$ for each $k \in K$ such that $p'_k = p_k$.

In Ausubel’s auction, Ausubel (2006), bidders’ values for bundles are assumed to be integer. The tâtonnement algorithm in Ausubel (2006) is designed to use this property of utility functions to move on the grid of integer price vectors, and to reach an integer-valued Walrasian equilibrium price vector. If the integer assumption on values is relaxed, more information on bidders’ values maybe needed to reach a Walrasian equilibrium price vector as shown by the example in Section 1.1. There is no straightforward way of modifying the tâtonnement algorithm so that it will always reach a Walrasian equilibrium when values are real. If bidders’ values for bundles are not integer, then there may not exist an integer-valued Walrasian equilibrium price vector, and the price adjustment procedure may not yield an efficient allocation when it ends. As Lemma 1 below shows, relaxing the integer-values restriction enriches the class of preferences of bidders.
A preference relation $\mathcal{R}$ defined on $X \times \mathbb{R}$ is representable if there exists a utility function $u : X \times \mathbb{R} \rightarrow \mathbb{R}$ such that for each $x, y \in X \times \mathbb{R}$,

$$x \mathcal{R} y \text{ if and only if } u(x) \geq u(y).$$

A preference relation $\mathcal{R}$ is said to be in $\mathcal{R}_D$ if and only if there exists a quasilinear utility function $u : X \times \mathbb{R} \rightarrow \mathbb{R}$ representing $\mathcal{R}$ such that $u(\cdot, t) = U(\cdot) + t$ where $U : X \rightarrow D$.

Lemma 1 shows that sets $\mathcal{R}_Z$, and $\mathcal{R}_\mathbb{R}$ are different.

Lemma 1. $\mathcal{R}_Z \subsetneq \mathcal{R}_\mathbb{R}$.

3 The Extension of Ausubel’s Auction

3.1 Dynamic Auction Game

The dynamic game is very similar to the one defined in Ausubel (2006), p.617 with the exception that the game presented here is a continuous game. There are $N$ bidders and an auctioneer with supply $S \in \mathbb{Z}_{+}^K$ of discrete goods to sell. The auction is designed as a continuous-time dynamic game. The time is a continuously increasing clock $t \in [0, \infty)$. At time $t$, each bidder $i$ observes history $H^i_t$ representing the prior play of the game. Let $x_i(p(t)) \subset X_i$ stand for the demand report bidder $i \in N$ makes at the price vector $p(t)$ reached at time $t$, and $x(p(t))$ stand for the demand profile reported at time $t$. History $H^i_t$ may, for example, consist of the complete history of demand profiles of all bidders, and price vectors. In that case $H^i_t = \{(x(p(s)), p(s)) | s \in [0, t)\}$.

The strategy $\sigma_i(H^i_t, t) \subset X_i$ of bidder $i$ is a set-valued function of history $H^i_t$ and time $t$ consisting of bundles of goods in $X_i$. Bidders are required to report their demand sets whenever they add a new bundle to their demand reports during the price adjustment.

In continuous-time games, there is a well-known problem of timing of strategies: If bidder $i$ reports a demand set at time $t$, bidder $j$ may want to report a demand set as a response to bidder $i$’s report at the soonest instant (see, for example, Simon and Stinchcombe (1989) for more discussion on this problem). Ausubel (2004), p.1465 uses a continuous-time auction in discrete homogeneous goods auction when values are symmetric and interdependent, and overcomes this problem by using rounds of bidding: if a bidder reduces the quantity he demands, then the auctioneer stops the clock and starts
rounds of bidding. Bidders are allowed to reduce their quantities at the current price. When there is no bidder who reduces his quantity at the current price, the auctioneer resumes the clock.

This approach, using bidding rounds, in my opinion, will unnecessarily complicate the auction design presented here. Instead, I will impose a restriction on strategy spaces of bidders to overcome this problem. The strategy space $\Sigma_i$ of each bidder $i$ consists of all such functions with the following restriction:

*Wait before reply.* At each time $t$, there exists $\epsilon^t_i > 0$ such that $x_i(\mathbf{p}(s)) \subset x_i(\mathbf{p}(t))$ for each $s \in (t, t + \epsilon^t_i)$.

Since bidders are required to report their demand sets whenever they add a new bundle to their demand sets, under the assumption stated above, at each time $t$ a bidder either reports his demand set or waits for some time before reporting it. Hence, the timing problem mentioned above does not appear. This *wait-before-reply* restriction is general enough to allow sincere bidding.

The auction terminates at time $T$ if there is a market clearing allocation of demands at time $T$. It is said to terminate at $T = \infty$ if there is an aggregate demand whose limit equals the aggregate supply of goods when $T \to \infty$. Then, the market clearing allocation of goods and the payments, which is explained below, is made. If the auction does not terminate, every bidder is assigned a payoff of $-\infty$.

Following Ausubel (2004) and Ausubel (2006), *ex post perfect equilibrium* is used in the dynamic game defined above:

The strategy profile $\{\sigma_i\}_{i \in N}$ constitutes an *ex post perfect equilibrium* if for every history $H^t_i$, and for every realization of private information $\{u_i\}_{i \in N}$, the profile of continuation strategies $\{\sigma_i(\cdot | H^t_i, u_i)\}_{i \in N}$ constitutes a Nash equilibrium of the game in which the realization of $\{u_i\}_{i \in N}$ is common knowledge.

### 3.2 The Price Adjustment Procedure

The description of the price adjustment procedure is similar to Ausubel’s tâtonnement algorithm given in Ausubel (2006), p.619. Unlike Ausubel’s tâtonnement algorithm, prices are continuously adjusted. Also, the auctioneer stops the clock whenever a bidder reports his demand set as he adds a new bundle to his demand set during the price adjustment.
Clock time \( t \in [0, \infty) \) of the price adjustment is called a step of the procedure either if \( t = 0 \) or if there is a bidder who reports his demand set at \( p(t) \).

1. The auctioneer sets the initial price vector \( p(0) = 0 \), and starting from \( t = 0 \) at each step asks each bidder \( i \) for his demand report \( Q_i(p(t)) \) at \( p(t) \).

2. All reports at \( p(t) \) are collected. Using these reports, the auctioneer determines the set of goods in excess demand \( E_+(p(t)) \), which is defined by choosing the smallest set satisfying equation 28 in Lemma 7. If bidders report truthfully and they have substitutes preferences (A.4), then by Proposition 3 this set is unique, i.e., all other sets satisfying equation 28 contain \( E_+(p(t)) \).

3. Prices of these goods in excess demand \( E_+(p(t)) \) are increased at the same rate continuously whereas prices of the rest of the goods remain the same. Observe that by quasilinearity of preferences, by the definition of minimal-cost-increase bundle given in equation 9, and by construction of the Lyapunov function given in equation 2 the set of goods in excess demand \( E_+(p(t)) \) does not change until the next step.

4. The extended ascending algorithm terminates when \( E_+(p(t)) = \emptyset \). By Propositions 4 and 5, when bidders have substitutes preferences and report truthfully, the algorithm terminates at the smallest Walrasian equilibrium price vector \( \mathbf{p} \). By Proposition 1 there exists a market clearing allocation at the equilibrium price vector reached, and this is the allocation of the auction. The extended descending algorithm works analogously.

3.3 Crediting and Debiting

The notion of “crediting and debiting,” which is introduced in Ausubel (2006) (see the example in Ausubel (2006), pp.607-608) is used in the extension proposed here, and works as follows: During the price adjustment, if opponents of bidder \( j \) lower the quantity they demand of a good in their bids, then it is credited to bidder \( i \) at price \( p(t) \). On the other hand, if opponents of bidder \( j \) rise the quantity they demand of a good, then it is debited from bidder \( j \) at price \( p(t) \). The auction ends, say at time \( T \in [0, \infty) \), whenever there
is a market clearing allocation in bidders' demand sets at $p(T)$. For each bidder credits and debits are added, monetary transfers are made, and goods are allocated. For an illustration of Ausubel’s auction, see the example in Ausubel (2006), pp. 606-607.

For each bidder $i \in N$, his payment is defined as

$$a_i(T) = p(0) \cdot [S - x_i(p(0))] - \int_0^T p(t) \cdot dx_i(p(t))$$

(1)

where $x_i(p(t))$ is a bundle bidder $i$ demands at price $p(t)$ and

$$x_{-i}(p(t)) = \sum_{j \neq i, j \in N} x_j(p(t)).$$

For a detailed discussion of this payment function see Ausubel (2006). In Section 4.3, I show that this payment function is well-defined, and path independent.

4 The Extended Tâtonnement Algorithms

4.1 Determining the Price Path

In the extended ascending and descending algorithms, at each price $p$, given the demand reports of bidders, the set of goods in excess demand is found using the function

$$L(p) = p \cdot S + \sum_{i \in N} V_i(p).$$

(2)

$L : \mathbb{R}_+^K \to \mathbb{R}$ is a Lyapunov function, and by envelope theorem, it is minimized at Walrasian equilibrium prices.

Proposition 1 below is from Ausubel (2005) and is valid without any change when bidders’ values for bundles are real rather than integer. Proposition 1 shows the relationship between the Walrasian equilibrium, the Lyapunov function, and the social surplus.

**Proposition 1** (Ausubel (2005)). Suppose that Assumptions A1 – A3 hold, and that a Walrasian equilibrium exists. Then, the set of Walrasian equilibrium price vectors equals the set of minimizers of $L(\cdot)$, and the set of Walrasian equilibria equals the set of all $(p^*, x^*)$ such that $p^* \in \mathbb{R}_+^K$ minimizes $L(\cdot)$ and $(x^*_i)_{i \in N}$ maximizes
\[ \sum_{i \in N} U_i(x_i) \text{ subject to } x_i \in X_i \text{ for all } i \in N, \]

and

\[ \sum_{i \in N} x_i \leq S. \]

**Corollary.** Suppose that Assumptions A1 – A4 hold. Then, the set of Walrasian equilibrium price vectors is a nonempty lattice and there exist the lowest and the highest Walrasian equilibrium price vectors, \( p \in \mathbb{R}^K_+ \) and \( \bar{p} \in \mathbb{R}^K_+ \), respectively.

The Corollary to Proposition 1 is also from Ausubel (2005), but integer properties of the highest and the lowest Walrasian price vectors are dropped as they are not necessarily true when bidders’ values for bundles are not restricted to integers. For proofs of Lemma 2, Proposition 1, and the Corollary to Proposition 1, see Ausubel (2005).

The Corollary to Proposition 1 implies that for any economy \( (\{u_i(\cdot)\}_{i \in N}, S) \), there exist Walrasian equilibrium price vectors \( p, \bar{p} \in \mathbb{R}_+^K \) such that if \( p^* \in \mathbb{R}_+^K \) is a Walrasian equilibrium price vector, then \( p \leq p^* \leq \bar{p} \).

### 4.2 Reaching Walrasian Equilibrium Prices

Theorem 1 below shows that the extended ascending algorithm and the extended descending algorithm stop after finitely many steps when values are real. In Ausubel’s auction, finite-step convergence is straightforward as the Lyapunov function in equation 2 is integer-valued and decreases by a positive integer amount at each integer price vector reached.

**Theorem 1.** Suppose that Assumptions A1 – A4 hold, and that bidders truthfully report their demands. Then, the extended ascending algorithm and the extended descending algorithm terminate in finite steps.

The proof consists of 3 parts: In the first part, the algorithm converges to a price vector. Then, in the second part, at each step of the price adjustment, each bidder who adds a bundle to his demand set at the current step has a bundle in his current demand set and a bundle in his demand set at the previous step such that one of these bundles can be constructed by adding
at most one unit of a good and removing at most one unit of another good from the other bundle. In the final part of the proof, using this feature of demand sets of bidders I show that at each step the total quantity of goods in excess demand strictly decreases.

Theorem 2 (Theorem 3) shows that if the initial prices are sufficiently low (high), then the extended ascending (descending) algorithm reaches the lowest (highest) Walrasian equilibrium price vector of the economy in finitely many steps.

Theorem 2. Suppose that Assumptions A1—A4 hold, and that bidders truthfully report their demands. Then, starting from any initial price vector of \( p(0) \in \mathbb{R}_+^K \) such that \( p(0) \leq p \), the extended ascending algorithm reaches the lowest Walrasian equilibrium price vector \( \overline{p} \) in finite steps.

The proof of the convergence of the extended descending algorithm to the highest Walrasian equilibrium price, Theorem 3 below, is analogous to the proof of Theorem 2.

Theorem 3. Suppose that Assumptions A1—A4 hold, and that bidders truthfully report their demands. Then, starting from any initial price vector of \( p(0) \in \mathbb{R}_+^K \) such that \( p(0) \geq \overline{p} \), the extended descending algorithm reaches the highest Walrasian equilibrium price vector \( \overline{p} \) in finite steps.

A price vector \( p' \in \mathbb{R}_+^K \) is a supporting price vector if there exists an allocation \( x' = (x'_i)_{i \in N} \) such that \( \sum_{i \in N} x_i \leq S \), and for each \( i \in N \), \( x'_i \in Q_i(p') \).

Theorem 2 imposes restrictions on the initial price vector. Theorem 4 shows that if these conditions are violated, then the extended ascending algorithm reaches in finite steps a supporting price vector whenever bidders report their demand sets truthfully. Theorem 4 is a generalization of Theorem 2 in Ausubel (2005) to economies with bidders who have real-valued utilities.

Theorem 4. Suppose that Assumptions A1—A4 hold, and that bidders truthfully report their demand. Let \( p(T) \) be a price vector the extended ascending algorithm reaches at \( T \in [0, \infty) \). Then, the extended ascending algorithm ends at \( p(T) \) in finite steps if and only if \( p(T) \) is a supporting price vector.
4.3 Truthful Bidding as an Ex Post Efficient Equilibrium

In the tâtonnement algorithm of Ausubel (2006), prices are adjusted in
discrete time and they take integer values. Ausubel (2006) converts the
tâtonnement algorithm to a continuous procedure by linearly increasing prices
between consecutive integer-valued price vectors, and uses the continuous-
time procedure to show that sincere bidding is an efficient equilibrium Ausubel’s
auction, and the procedure yields Walrasian equilibrium prices. The ex-
tended tâtonnement algorithm is a continuous-time procedure. Observe
that the extended tâtonnement algorithm and the continuous version of the
tâtonnement algorithm follow the same path if bidders’ values for bundles are
integer, the initial price vector is integer-valued, and the extended ascending
algorithm has a unit rate of change. In that case, the number of steps in the
extended algorithm will be less than or equal to the number of times demand
reports collected in Ausubel’s auction as bidders may not add new bundles
at some of the integer price vectors reached.

The price adjustment process can be written as
\[
\frac{dp^k(t)}{dt} = \begin{cases} 
  c_{E(p(t))} & \text{if } k \in E(p(t)) \\
  0 & \text{if } k \notin E(p(t))
\end{cases}
\]
for all \(t\) except when \(t\) is a step, and
\[
E(p(t)) = E_+(p(t)) \text{ and } c_{E_+(p(t))} > 0
\]
for the extended ascending algorithm, and
\[
E(p(t)) = E_-(p(t)) \text{ and } c_{E_-(p(t))} < 0
\]
for the extended descending algorithm.

Next theorem is the main result of the paper. Mandatory participation
is assumed as, because of the payment system, a bidder may have a negative
payoff at the end of the auction for some initial price vector (see Ausubel

**Theorem 5.** Suppose that Assumptions A1—A4 hold, and there is mandatory
participation:

i. Truthful demand reports by every bidder constitute an ex post efficient
equilibrium of the auction game, and
ii. Starting at any sufficiently small (large) price vector \( p(0) \in \mathbb{R}_+^K \) if bidders truthfully report their demands, then the extended ascending (descending) algorithm reaches a Walrasian equilibrium price vector in finitely many steps.

Proof of this theorem is very similar to the proofs of the corresponding results for integer values in Theorem 2’ (see Ausubel (2006), p. 620), which is explained below. By Lemma 1 in Ausubel (2006), if \( p(\cdot) \) is continuous and for each \( j \in N, j \neq i \), any \( k \in K, x_j^k(\cdot) \), amount of good \( k \) demanded by bidder \( j \), is of bounded variation, then the payment function \( a_i(T) \) given in equation 1 is well-defined. Moreover, if \( p(\cdot) \) is also a piecewise smooth function from \([0, T]\) to \( \mathbb{R}_+^K \), and for each \( j \in N, j \neq i \), \( U_j(\cdot) \) is a concave and continuous function, then by Lemma 2 in Ausubel (2006)

\[
a_i(T) = p(0) \cdot [S - x_{-i}(p(0))] - \sum_{j \neq i, j \in N} [U_j(x_j(p(T))) - U_j(x_j(p(0)))] .
\]

Observe that the price path induced by the extended ascending (descending) algorithm is piecewise linear and continuous. Therefore, since substitutes preferences imply concave utility functions (see Ausubel (2006), p.617, and Milgrom and Strulovici (2009)), the conditions of Lemmas 1 and 2 in Ausubel (2006) hold. So, by Lemmas 1 and 2 in Ausubel (2006), payment function given in equation 1 is well-defined and path-independent, and the equality 3 holds for the extended tâtonnement algorithms. Hence, Theorems 1’ and 2’ in Ausubel (2006) hold literally (see the discussion on p.620 Ausubel (2006)) for the extension of the Ausubel auction proposed in this paper. As a result, starting at a sufficiently low (high) price vector, the truthful bidding constitutes an efficient equilibrium in the extension of Ausubel’s auction, and the extended ascending (descending) algorithm terminates at a Walrasian equilibrium price vector in finitely many steps.

5 Conclusion

In this paper, I extended the ascending and the descending price adjustment procedures proposed in Ausubel (2006) to real-valued quasilinear utility functions. I show that these extended procedures converge to a Walrasian equilibrium price vector in finite steps. Unlike the tâtonnement algorithm in Ausubel (2006), the extended tâtonnement algorithm does not require any information on bidders’ values.
Appendix

Proof of Lemma 1. \( \mathcal{R}_Z \subset \mathcal{R}_R \) is trivial. I will show that there exists \( \mathcal{R} \in \mathcal{R}_R \) such that \( \mathcal{R} \not\in \mathcal{R}_Z \). Let \( \mathcal{R} \in \mathcal{R}_R \) be such that there exist \( x, y \in X \) and \( t \in \mathbb{R} \setminus \mathbb{Z} \) such that

\[
(x, t) \sim_R (y, 0).
\] (4)

By the definition of \( \mathcal{R}_R \), there exists \( u(x, t) = U(y) + t \) representing \( \mathcal{R} \), where \( U : X \rightarrow \mathbb{R} \). Equation 4 implies that

\[
u(x, t) = u(x, t),
\]
(5)

Now, assume on the contrary that, there exists \( \tilde{u}(x, t) = \tilde{U}(y) + t \) representing \( \mathcal{R} \) such that \( \tilde{U} : X \rightarrow \mathbb{Z} \). Equation 4 implies that \( \tilde{u}(x, t) = \tilde{u}(y, 0) \). So, \( t = \tilde{U}(y) - \tilde{U}(x) \). Equation 5 implies that \( t = \tilde{U}(y) - \tilde{U}(x) \in \mathbb{R} \setminus \mathbb{Z} \), a contradiction to equation 6. Hence, \( \mathcal{R}_Z \not\subset \mathcal{R}_R \).

The following Lemma from Ausubel (2005) shows important properties of the Lyapunov function \( L(\cdot) \) of equation 2.

A function \( L : \mathbb{R}_+^K \rightarrow \mathbb{R} \) is \textit{submodular} if for each \( p, p' \in \mathbb{R}_+^K \)

\[
L(p \wedge p') + L(p \vee p') \leq L(p) + L(p').
\]

Lemma 2 (Ausubel (2005)). \textit{Under the gross substitutes assumption A4, the Lyapunov function \( L(\cdot) \) of equation 2 is a submodular and convex function.}

Lemma 3 shows that as a set of prices are increased (decreased) in the extended ascending (descending) algorithm, if a bidder removes a bundle from his demand set, he does so immediately, i.e. as soon as prices start to increase. Lemma 3 is used in the proofs of Lemma 6 and Theorem 1

Lemma 3. Suppose that Assumptions A1 – A2 hold. Let \( i \in N, x_i, x'_i \in X_i \), and \( p \in \mathbb{R}_+^K \) be such that \( U_i(x_i) - p \cdot x_i = U_i(x'_i) - p \cdot x'_i \).

If there exist \( \Delta \in \mathbb{R}_+^K \) and \( \lambda' > 0 \) such that

\[
U_i(x_i) - (p + \lambda' \Delta) \cdot x_i > U_i(x'_i) - (p + \lambda' \Delta) \cdot x'_i,
\]
then for each \( \lambda > 0 \)

\[
U_i(x_i) - (p \mp \lambda \Delta) \cdot x_i > U_i(x'_i) - (p \mp \lambda \Delta) \cdot x'_i,
\]

and if there exist \( \Delta \in \mathbb{R}_+^K, \Delta \neq 0 \) and \( \lambda' > 0 \) such that

\[
U_i(x_i) - (p \mp \lambda' \Delta) \cdot x_i = U_i(x'_i) - (p \mp \lambda' \Delta) \cdot x'_i,
\]

then for each \( \lambda > 0 \)

\[
U_i(x_i) - (p \mp \lambda \Delta) \cdot x_i = U_i(x'_i) - (p \mp \lambda \Delta) \cdot x'_i.
\]

Proof. Suppose that \( p \in \mathbb{R}_+^K, \), and \( x_i, x'_i \in X_i \) such that

\[
U_i(x_i) - p \cdot x_i = U_i(x'_i) - p \cdot x'_i. \tag{7}
\]

Then, by equation 7, for each \( \Delta \geq 0 \) and for each \( \lambda > 0 \),

\[
U_i(x_i) - (p \mp \lambda \Delta) \cdot x_i > U_i(x'_i) - (p \mp \lambda \Delta) \cdot x'_i
\]

if and only if

\[
0 > \mp \lambda \Delta \cdot (x_i - x'_i). \tag{8}
\]

Observe that inequality 8 holds for some \( \lambda'' > 0 \) if and only if it holds for all \( \lambda > 0 \).

If there exist \( \Delta \in \mathbb{R}_+^K, \Delta \neq 0 \) and \( \lambda' > 0 \) such that

\[
U_i(x_i) - (p \mp \lambda' \Delta) \cdot x_i = U_i(x'_i) - (p \mp \lambda' \Delta) \cdot x'_i,
\]

then by equation 7,

\[
\mp \lambda' \Delta \cdot (x_i - x'_i) = 0.
\]

Thus, for each \( \lambda > 0 \),

\[
\mp \lambda \Delta \cdot (x_i - x'_i) = 0.
\]

Hence, the result follows. \( \square \)

Suppose that a bidder strictly prefers one bundle to another. If prices are increased (decreased) linearly, then there are two possibilities: either he will keep preferring one to the other, or he will be indifferent between them at a unique price and he will reverse his preferences over these bundles as prices continue to linearly increase (decrease). This is formally stated in Lemma 4, which is used in the proofs of Lemmas 5 and 6, and Proposition 2.
Lemma 4. Suppose that Assumptions A1$-A2$ hold. Let $i \in N$, $x_i, x_i' \in X_i$, $p \in \mathbb{R}^K_+$, and $\Delta \in \mathbb{R}^K_+$ be such that $U_i(x_i) - (p + \Delta) \cdot x_i > U_i(x_i') - (p + \Delta) \cdot x_i'$. Then, one of the following holds:

1. There exists a unique $\lambda_{x_i'} > 0$ such that

$$U_i(x_i) - (p + \lambda_{x_i'} \Delta) \cdot x_i = U_i(x_i') - (p + \lambda_{x_i'} \Delta) \cdot x_i',$$

and for each $\lambda \in (0, \lambda_{x_i'})$

$$U_i(x_i) - (p + \lambda \Delta) \cdot x_i > U_i(x_i') - (p + \lambda \Delta) \cdot x_i',$$

and for each $\lambda > \lambda_{x_i'}$

$$U_i(x_i) - (p + \lambda \Delta) \cdot x_i < U_i(x_i') - (p + \lambda \Delta) \cdot x_i'.$$

2. For all $\lambda > 0$, $U_i(x_i) - (p + \lambda \Delta) \cdot x_i > U_i(x_i') - (p + \lambda \Delta) \cdot x_i'$.

Proof. If $\nabla \cdot (x_i - x_i') > 0$ then $U_i(x_i) - (p + \lambda_{x_i'} \Delta) \cdot x_i = U_i(x_i') - (p + \lambda_{x_i'} \Delta) \cdot x_i'$ for $\lambda_{x_i'} = \frac{U_i(x_i) - p \cdot x_i - U_i(x_i') - p \cdot x_i'}{\Delta(x_i - x_i')}$. If $\nabla \cdot (x_i - x_i') \leq 0$ then $U_i(x_i) - (p + \lambda \Delta) \cdot x_i > U_i(x_i') - (p + \lambda \Delta) \cdot x_i'$ for all $\lambda > 0$.

Suppose that a bidder $i \in N$ announces his demand set at $Q_i(p) \subset X_i$ at price vector $p \in \mathbb{R}^K_+$. I will now show how the auctioneer can determine bundles that will stay in bidder $i$'s demand set when prices of a subset of goods are all slightly increased (or decreased).

For each $p, \Delta \in \mathbb{R}^K_+$, a minimal-cost-increase bundle $\tilde{x}_i(p, \Delta) \in Q_i(p)$ is a bundle bidder $i$ demands at $p$ that has the lowest cost increase when prices increase from $p$ to $p + \Delta$, i.e.

$$\tilde{x}_i(p, \Delta) \in \arg \min_{x \in Q_i(p)} \{\Delta \cdot x\}. \tag{9}$$

Analogously, a maximal-cost-decrease bundle $\tilde{y}_i(p, \Delta) \in Q_i(p)$ is a bundle bidder $i$ demands at $p$ that has the highest cost decrease when prices decrease from $p$ to $p - \Delta$, i.e.

$$\tilde{y}_i(p, \Delta) \in \arg \max_{x \in Q_i(p)} \{\Delta \cdot x\}. \tag{10}$$

19
Define
\[ \delta_i(p) = \frac{1}{2} \inf_{y \in X_i} \frac{1}{X_i} \sum_{k \in K} (U_i(\tilde{x}_i(p, \Delta)) - p \cdot \tilde{x}_i(p, \Delta)) - (U_i(y) - p \cdot y). \]

If \( \delta_i(p) \) is finite, then \( \delta_i(p) \) is the difference between the highest and the second highest utility levels bidder \( i \) can achieve from bundles in \( X_i \) at price \( p \) as \( X_i \) is finite. Hence, for each \( p, \Delta \in \mathbb{R}_+^K \) either
\[ \delta_i(p) = +\infty \]
or
\[ \delta_i(p) > 0. \]

Note that \( \delta_i(p) \) is infinite if and only if bidder \( i \) is indifferent among all bundles at prices \( p \).

Construct \( \delta(p) \) as follows:
\[ \delta(p) = \min_{i \in N} \delta_i(p) \] (11)

Since there are finitely many bidders in \( N \),
\[ \delta(p) > 0. \]

By the definition of \( \delta(p) \), for each \( p \in \mathbb{R}_+^+ \), for each \( i \in N \), for each \( x \in Q_i(p) \), and for each \( y \in X_i \setminus Q_i(p) \)
\[ (U_i(x) - p \cdot x) - (U_i(y) - p \cdot y) > \delta(p). \]

Proposition 2 shows that if prices are increased no more than \( \delta(p) \) at \( p \), then bidders do not add any bundles to their demand sets during this price increase. Proposition 2 is used in the proofs of Proposition 4, the Corollary to Proposition 3, and Lemmas 5, 6 and 7. Proposition 2 is a generalization of Proposition 2 in Ausubel (2006), p.625 without the monotonicity (A.3) and the substitutes (A.4) assumptions.

For each \( \delta \in \mathbb{R} \), let \( \delta^K = (\delta^k)_{k \in K} \) denote a \( K \)-dimensional vector such that \( \delta^k = \delta \) for all \( k \in K \). Proposition 2 shows that the auctioneer can compute each bidder’s demand set for all prices in the \( K \)-dimensional \( \delta(p) \)-wide cube above (below) price vector \( p \) using that bidder’s demand set at \( p \).
Proposition 2. Suppose that Assumptions A1 – A2 hold. For each $i \in N$, for each $p \in \mathbb{R}^K_+$, for each $\Delta \in \mathbb{R}^K_+$, and for each $\lambda \in \mathbb{R}_{++}$ such that $0 \leq \Delta \leq \delta(p)^K$, and $0 \leq \lambda \Delta \leq \delta(p)^K$

$$Q_i(p + \Delta) = \{\tilde{x}_i(p, \Delta) \in Q_i(p)\},$$
and

$$Q_i(p - \lambda \Delta) = \{\tilde{y}_i(p, \Delta) \in Q_i(p)\}.$$

Proof. Let $i \in N$, $p \in \mathbb{R}^K_+$, and $\Delta \in \mathbb{R}^K_+$ such that $0 \leq \Delta \leq \delta(p)^K$. Let $y \in X_i$.

Claim. If $y \not\in \{\tilde{x}_i(p, \Delta) \in Q_i(p)\}$, then for each $\lambda \in \mathbb{R}_{++}$ such that $0 \leq \lambda \Delta \leq \delta(p)^K$

$$U_i(\tilde{x}_i(p, \Delta)) - (p + \lambda \Delta) \cdot \tilde{x}_i(p, \Delta) > U_i(y) - (p + \lambda \Delta) \cdot y. \quad (12)$$

Proof of the Claim. There are two cases to consider:

Case 1. If $y \not\in \{\tilde{x}_i(p, \Delta) \in Q_i(p)\}$, then for each $\lambda \in \mathbb{R}_{++}$ such that $0 \leq \lambda \Delta \leq \delta(p)^K$

$$U_i(\tilde{x}_i(p, \Delta)) - (p + \lambda \Delta) \cdot \tilde{x}_i(p, \Delta) > U_i(y) - (p + \lambda \Delta) \cdot y.$$

Case 2. If $y \not\in Q_i(p)$, then

$$U_i(\tilde{x}_i(p, \Delta)) - p \cdot \tilde{x}_i(p, \Delta) > U_i(y) - p \cdot y.$$

Let $\lambda \in \mathbb{R}_{++}$ such that $0 \leq \lambda \Delta \leq \delta(p)^K$. Observe that as $\Delta \geq 0$, and $\tilde{x}_i(p, \Delta), y \in X_i$

$$\lambda \Delta \cdot (\tilde{x}_i(p, \Delta) - y) \leq \lambda \Delta \cdot (\bar{x}_k^i)_{k \in K} = \lambda \sum_{k \in K} (\Delta^k \bar{x}_k^i)$$

Hence,

$$\lambda \Delta \cdot (\tilde{x}_i(p, \Delta) - y) \leq \lambda \sum_{k \in K} (\Delta^k \bar{x}_k^i). \quad (13)$$

If there does not exist $\lambda_y > 0$ such that

$$U_i(\tilde{x}_i(p, \Delta)) - (p + \lambda \Delta) \cdot \tilde{x}_i(p, \Delta) > U_i(y) - (p + \lambda \Delta) \cdot y,$$
then by Lemma 4
\[ U_i(\tilde{x}_i(p, \Delta)) - (p + \lambda \Delta) \cdot \tilde{x}_i(p, \Delta) > U_i(y) - (p + \lambda \Delta) \cdot y. \]  \tag{14} 

If there exists such \( \lambda_y > 0 \), then, as
\[(U_i(\tilde{x}_i(p, \Delta)) - p \cdot \tilde{x}_i(p, \Delta)) - (U_i(y) - p \cdot y) = (\tilde{x}_i(p, \Delta) - y) \cdot \lambda_y \Delta, \]
and by the definition of \( \delta(p) \), for each \( k \in K \)
\[\lambda \Delta^k \leq \delta(p) < \lambda_y \Delta \cdot (\tilde{x}_i(p, \Delta) - y) \frac{1}{\sum_{k \in K} x_i^k}.\]
Thus, by multiplying with \( x_i^k \)
\[\lambda \Delta^k x_i^k < \lambda_y \Delta \cdot (\tilde{x}_i(p, \Delta) - y) \frac{x_i^k}{\sum_{k \in K} x_i^k},\]
and summing over \( k \in K \)
\[\lambda \sum_{k \in K} \Delta^k x_i^k < \lambda_y \Delta \cdot (\tilde{x}_i(p, \Delta) - y). \]  \tag{15} 

Inequalities 13 and 15 imply that
\[\lambda \Delta \cdot (\tilde{x}_i(p, \Delta) - y) < \lambda_y \Delta \cdot (\tilde{x}_i(p, \Delta) - y).\]
So, by the definition of \( \lambda_y \)
\[\lambda \Delta \cdot (\tilde{x}_i(p, \Delta) - y) < \left( U_i(\tilde{x}_i(p, \Delta)) - p \cdot \tilde{x}_i(p, \Delta) \right) - \left( U_i(y) - p \cdot y \right). \]  \tag{16} 

Hence,
\[U_i(\tilde{x}_i(p, \Delta)) - (p + \lambda \Delta) \cdot \tilde{x}_i(p, \Delta) > U_i(y) - (p + \lambda \Delta) \cdot y. \]  \tag{17} 
This completes the proof of the Claim. \( \square \)

By the definition of \( \tilde{x}_i(p, \Delta) \), for all minimal-cost-increase bundles \( \tilde{x}_i(p, \Delta) \) and \( \bar{x}_i(p, \Delta) \), and for each \( \lambda > 0, \)
\[U_i(\bar{x}_i' (p, \Delta)) - (p + \lambda \Delta) \cdot \bar{x}_i'(p, \Delta) = U_i(\bar{x}_i(p, \Delta)) - (p + \lambda \Delta) \cdot \bar{x}_i(p, \Delta). \]  \tag{18} 

Equations 12, 14, 18 and 17 imply that
\[Q_i(p + \lambda \Delta) = \{ \tilde{x}_i(p, \Delta) \in Q_i(p) \}.\]
Analogously it can be shown that
\[Q_i(p - \lambda \Delta) = \{ \bar{y}_i(p, \Delta) \in Q_i(p) \}.\] \( \square \)
For each \( \delta \in \mathbb{R} \), and for each subset of goods \( E \subset K \), let \( \delta^E = (\delta^k)_{k \in K} \) denote a \( K \)-dimensional vector such that \( \delta^k = 0 \) if \( k \notin E \), and \( \delta^k = \delta \) if \( k \in E \).

As prices change, bidders’ demand sets change: a bidder may remove a bundle from his demand set, or add a bundle to his demand set. Lemma 5 shows a relationship between bundles in the demand set and those added during the price adjustment. Lemma 5 is used in the proofs of Lemma 6, and Theorem 1.

**Lemma 5.** Suppose that Assumptions A1 – A4 hold. Let \( i \in N \), \( \mathbf{p}, \mathbf{p}' \in \mathbb{R}^K_+ \) such that \( \mathbf{p} \leq \mathbf{p}' \),
\[
Q_i(\mathbf{p}') \setminus Q_i(\mathbf{p}) \neq \emptyset,
\]
and
\[
Q_i(\mathbf{p}') \cap Q_i(\mathbf{p}) \neq \emptyset. \tag{19}
\]
Then, there exist
\[
x'_i \in Q_i(\mathbf{p}') \setminus Q_i(\mathbf{p}),
\]
and
\[
x_i \in Q_i(\mathbf{p}') \cap Q_i(\mathbf{p})
\]
such that
\[
\#(x_i \setminus x'_i) = 1 \text{ and } \#(x'_i \setminus x_i) \leq 1.
\]
Moreover, there exists a unique \( k \in K \) such that
\[
x_i^k = x'_i^k - 1.
\]
Also,
\[
p_k < p'_k.
\]
On the other hand, if there exists \( k' \in K \) such that \( x_i^{k'} = x_i^{k'} + 1 \), then
\[
p_{k'} - p_{k'} < p'_k - p_k.
\]
If there exists \( E \subset K \) and \( \delta \in \mathbb{R}^{++} \) such that \( \mathbf{p}' = \mathbf{p} + \delta^E \), then
\[
k \in E,
\]
and
\[
k' \notin E \text{ if there exists } k' \in K \text{ such that } x_i^{k'} = x_i^{k'} + 1.
\]
Proof. Define $\Delta = p' - p$. By Lemma 4, equation 19 implies that there does not exist $\lambda \in (0, 1)$ such that

$$Q_i(p + \lambda \Delta) \setminus Q_i(p) \neq \emptyset,$$

i.e. as prices are increased linearly from $p$ to $p'$, $p'$ is the first price vector reached at which a new bundle is added to the demand set.

Construction of $x'_i$. Let $x'_i \in Q_i(p') \setminus Q_i(p)$ such that for each $y \in Q_i(p') \setminus Q_i(p)$

$$U_i(x'_i) - p \cdot x'_i \geq U_i(y) - p \cdot y.$$

$x'_i$ is a bundle bidder $i$ adds to his demand set at $p'$ such that bidder $i$ prefers this bundle at $p$ to all other bundles added at $p'$. Such $x'_i$ exists because $X_i$ is finite.

Construction of $x_i$. By Lemma 4, for each $y \in Q_i(p') \setminus Q_i(p)$, and for each $\lambda \in [0, 1]$

$$U_i(x'_i) - (p + \lambda \Delta) \cdot x'_i \geq U_i(y) - (p + \lambda \Delta) \cdot y.$$

As $X_i$ is finite, by Lemma 4, there exists $\lambda' \in (0, 1)$ such that for each $\lambda \in (\lambda', 1)$, and $y \in X_i$, if

$$U_i(y) - (p + \lambda \Delta) \cdot y > U_i(x'_i) - (p + \lambda \Delta) \cdot x'_i$$

then

$$y \in Q_i(p + \lambda \Delta).$$

In words, there exists a price $p + \lambda \Delta$ such that at all prices reached between $p + \lambda \Delta$ and $p'$, there is no bundle both strictly preferred to $x'_i$ and not in bidder $i$’s demand set. Hence, by the single-improvement property (implied by Assumptions the gross substitutes $A4$ and monotonicity $A3$, see Lemma 2 in Gul and Stacchetti (1999)), there exists

$$x_i \in Q_i(p + \lambda \Delta)$$

such that

$$\#(x_i \setminus x'_i) \leq 1 \text{ and } \#(x'_i \setminus x_i) \leq 1.$$

Since the first new bundle is added at $p'$ to bidder $i$’s demand set as prices are increased linearly from $p$ to $p'$, by Proposition 2,

$$Q_i(p + \lambda \Delta) = Q_i(p) \cap Q_i(p').$$
Therefore,

$$x_i \in Q_i(p) \cap Q_i(p'),$$

and

$$\#(x_i \setminus x'_i) \leq 1 \text{ and } \#(x'_i \setminus x_i) \leq 1.$$  

(20)

So, $x_i$ is a bundle in bidder $i$'s demand sets at all prices reached between $p$ and $p'$ such that $x'_i$ can be constructed by adding at most a unit of a good and removing at most a unit of another good to $x_i$.

As $U_i(x'_i) - (p + \Delta) \cdot x'_i = U_i(x_i) - (p + \Delta) \cdot x_i$,

$$-(U_i(x'_i) - p \cdot x'_i) + (U_i(x_i) - p \cdot x_i) = (x_i - x'_i) \cdot \Delta.$$

Since $x_i \in Q_i(p)$ and $x'_i \notin Q_i(p)$,

$$(x_i - x'_i) \cdot \Delta > 0.$$

Therefore, by equations 20, there exists a unique $k \in K$ such that $x'^k_i = x^k_i - 1$. Observe that $\Delta_k = p'_k - p_k > 0$. Moreover, if there exists $k' \in K$ such that $x'^k_i = x'^{k'}_i + 1$, then $\Delta_k - \Delta_{k'} > 0$ implying $p'_k - p_k < p'_k - p_k$. Otherwise, $\Delta_k > 0$ implying $p_k < p'_k$. The rest of the proof follows trivially. \qed

Lemma 6 explains the relationship between each bidder’s demand set on the corners of $\delta(p)$-wide cube above (below) price vector $p$ and his demand sets on the edges of that cube. Lemma 6 is used in the proof of Proposition 3.

\textbf{Lemma 6.} Suppose that Assumptions A1 – A4 hold. For each $p \in \mathbb{R}_+^K$, for each $\delta \in [0, \delta(p)]$, for each $E \subset K$, for each $i \in N$, and for each $\Delta \geq 0$ such that for each $k \in E$ $\Delta^k = 0$, and for each $\lambda \in \mathbb{R}_+$ such that $0 \leq \lambda \Delta \leq \delta^K$

$$\{\bar{x}_i(p + \delta^E, \Delta) \in Q_i(p + \delta^E)\} = Q_i(p + \delta^E + \lambda \Delta),$$  

(21)

and

$$\{\bar{y}_i(p - \delta^E, \Delta) \in Q_i(p - \delta^E)\} = Q_i(p - \delta^E - \lambda \Delta).$$

\textit{Proof.} Consider the equation 21. Suppose on the contrary that there exist $p \in \mathbb{R}_+^K$, $\delta \in [0, \delta(p)]$, $E \subset K$, $i \in N$, $\lambda \in \mathbb{R}_+$, $\Delta \in \mathbb{R}_+^K$ and $\bar{x}_i(p + \delta^E, \Delta)$ such that $0 \leq \lambda \Delta \leq \delta^K$, and for each $k \in E$ $\Delta^k = 0$, and equation 21 does not hold. Then, either there exists $\tilde{x}_i(p + \delta^E, \Delta) \in Q_i(p + \delta^E)$ such that

$$\tilde{x}_i(p + \delta^E, \Delta) \notin Q_i(p + \delta^E + \lambda \Delta),$$  

(22)
or there exists

\[ x_i \in Q_i(p + \delta E + \lambda \Delta) \setminus \{ \tilde{x}_i(p + \delta E, \Delta) \in Q_i(p + \delta E) \}. \]  

(23)

Suppose that equation 22 holds. If \( \delta = 0 \) or \( \lambda \Delta = 0 \), then equation 22 contradicts to the definition of \( \tilde{x}_i(\cdot, \cdot) \). So, suppose that \( \delta > 0 \) and \( \lambda \Delta \geq 0 \). By Lemma 3, equation 22 implies that

\[ \{ \tilde{x}_i(p + \delta E, \Delta) \in Q_i(p + \delta E) \cap Q_i(p + \delta E + \lambda \Delta) = \emptyset, \]

which means that all minimal cost bundles are dropped from the demand set at the same price vector which is between \( p + \delta E \) and \( p + \delta E + \lambda \Delta \). Hence, by Lemma 3 again,

\[ Q_i(p + \delta E) \cap Q_i(p + \delta E + \lambda \Delta) = \emptyset, \]

which implies equation 23. So, it is sufficient to show that equation 23 results in a contradiction.

Define

\[ \lambda = \min \{ \lambda' \in \mathbb{R}^+ : \text{there exists } x'_i \in X_i \setminus Q_i(p + \delta E) \text{ such that } \]

\[ U_i(x'_i) - (p + \delta E + \lambda' \Delta) \cdot x'_i = U_i(\tilde{x}_i(p + \delta E, \Delta)) - (p + \delta E + \lambda' \Delta) \cdot \tilde{x}_i(p + \delta E, \Delta) \}. \]

Note that \( p + \delta E + \lambda \Delta \) is the first price vector reached at which bidder \( i \) adds a bundle to his demand set as prices increase linearly from \( p + \delta E \) to \( p + \delta E + \lambda \Delta \). Let \( \bar{x}_i \in X_i \) be a bundle which bidder \( i \) adds to his demand set at \( p + \delta E + \lambda \Delta \) such that \( \bar{x}_i \) is preferred at all prices between \( p + \delta E \) and \( p + \delta E + \lambda \Delta \) to all other bundles which bidder \( i \) adds to his demand set at \( p + \delta E + \lambda \Delta \). Since \( X_i \) is finite, by Lemma 4 and equation 23, \( \lambda \) is well-defined, and \( \lambda > 0 \). Observe that

\[ \lambda < \lambda, \]  

(24)

and hence \( 0 \leq \lambda \Delta \ll \delta K \). So, \( \bar{x}_i \in X_i \) is such that

\[ U_i(\bar{x}_i) - (p + \delta E + \lambda \Delta) \cdot \bar{x}_i = U_i(\tilde{x}_i(p + \delta E, \Delta)) - (p + \delta E + \lambda \Delta) \cdot \tilde{x}_i(p + \delta E, \Delta), \]

(25)

and for each \( \lambda' \in [0, \lambda] \), and for each \( x_i \in Q_i(p + \delta E + \lambda \Delta) \setminus Q_i(p + \delta E) \)

\[ U_i(\bar{x}_i) - (p + \delta E + \lambda' \Delta) \cdot \bar{x}_i \geq U_i(x_i) - (p + \delta E + \lambda' \Delta) \cdot x_i. \]
Note that \( x_i \notin Q_i(p + \delta^E) \), and by Lemma 5, there exists 
\[
\tilde{x}_i(p + \delta^E, \Delta) \in Q_i(p + \delta^E + \lambda \Delta)
\]
such that 
\[
\#(\tilde{x}_i(p + \delta^E, \Delta) \setminus x_i) = 1
\]
and 
\[
\#(x_i \setminus \tilde{x}_i(p + \delta^E, \Delta)) \leq 1.
\]
By Proposition 2, 
\[
x_i, \tilde{x}_i(p + \delta^E, \Delta) \in Q_i(p).
\]
Hence, 
\[
U_i(x_i) - p \cdot x_i = U_i(\tilde{x}_i(p + \delta^E, \Delta)) - p \cdot \tilde{x}_i(p + \delta^E, \Delta),
\]
and by equation 25, 
\[
(\tilde{x}_i(p + \delta^E, \Delta) - x_i)(\delta^E + \lambda \Delta) = 0.
\]
Then, either 
\[
\lambda \Delta^k = 0 \text{ for some } k \in K,
\]
a contradiction to Lemma 5 for prices \( p + \delta^E \) and \( p + \delta^E + \lambda \Delta \), or 
\[
\lambda \Delta^k - \delta = 0 \text{ for some } k \in K,
\]
a contradiction to equation 24 as \( \lambda \Delta \leq \delta^K \). An analogous argument can be made for \( \tilde{y}_i(p - \delta^E, \Delta) \).

Proposition 3 shows that for each price vector there exists a unique set of goods which determine the direction in which prices increase (decrease) in the extended algorithm. Proposition 3 is a straightforward generalization of Proposition 3 in Ausubel (2005), p.17, and it is used in the proofs of Propositions 4, 5, and Lemma 7. The Corollary to Proposition 3 is used in the proof of Theorem 1.

A minimal minimizer \( p_+(\cdot) \) is such that for each \( p \in \mathbb{R}_+^K \),
\[
p_+(p) \in \arg \min_{\bar{p} \in \{p + \lambda \Delta : 0 \leq \Delta \leq \delta(p)^K\}} \{L(\bar{p})\}, \tag{26}
\]
such that for each \( p' \in \mathbb{R}_+^K \) if \( p \leq p' \leq p_+(p) \), then \( L(p') > L(p_+(p)) \).
So, a minimal minimizer $p_+(p)$ is a minimizer of the Lyapunov function of equation 2 in $\{p + \Delta : 0 \leq \Delta \leq \delta(p)^K\}$ and this set does not contain any price vector which is less than or equal to $p_+(p)$ in every coordinate, and a minimizer of the Lyapunov function in that set.

Similarly, for each $p \in \mathbb{R}_+^K$, a maximal minimizer $p_-(\cdot)$ is defined as

$$p_-(p) \in \arg\min_{\tilde{p} \in \{p - \Delta : 0 \leq \Delta \leq \delta(p)^K\}} \{L(\tilde{p})\},$$

such that for each $p' \in \mathbb{R}_+^K$ if $p \geq p' \geq p_-(p)$, then $L(p') > L(p_-(p)).$

**Proposition 3.** Suppose that Assumptions A1 – A4 hold, and that bidders truthfully report their demand. Then, at each price $p \in \mathbb{R}_+^K$, and at each $\delta \in [0, \delta(p)]$, there exist a unique minimal minimizer $p_+(p)$, a unique maximal minimizer $p_-(p)$, and sets of goods $E_+(p), E_-(p) \subset K$ such that

$$p_+(p) = p + \delta E_+(p) \quad (27)$$

and

$$p_-(p) = p - \delta E_-(p).$$

**Corollary.** Suppose that Assumptions A1 – A4 hold, and that bidders truthfully report their demand. Then, for each step $t$ of the extended ascending (descending) algorithm, if the procedure does not terminate at $t$, then there exists a step $t' > t$ such that there is no step $\hat{t}$ such that $\hat{t} \in (t, t').$

**Proof of the Corollary.** Let $t \in [0, \infty)$ be a step of the algorithm, and $p(t)$ be the corresponding price vector. If $t$ is not the last step, then, by Propositions 2 and 3, there exist $\delta(p) > 0$ and $E \subset K$ such that no bidder adds any bundle at any price reached between $p(t)$ and $p(\hat{t}) = p(t) + \delta E$. Hence, there is no step between $t$ and $\hat{t}$. Let $t' = \inf_{t'' > t \text{ and } t'' \text{ is a step}} t''$. Observe that $t' > t$, and $t'$ is a step.

Proposition 3 is proven by replacing 1 with $\delta$ in the proof of Proposition 3 in Ausubel (2005), p.17.

The goods in $E_+(p)$ are called the goods in excess demand at price $p$. This notion of excess demand is different from the classical notion of excess demand. Gul and Stacchetti (2000) discuss the need for a different notion of excess demand because of the problems arising during the price adjustment.
when goods are discrete. Gul and Stacchetti (2000) point out to the distinction between the level of excess demand and the sum of quantities of goods in excess demand in discrete goods. For example, two different goods can be in excess demand but the level of excess demand may only be 1. The set of goods in excess demand according to the classical notion and the one here coincide if the demand correspondences are assumed to be single-valued. In other words, the same set of goods will be in excess demand.

Lemma 7 gives a simple method for determining the set of goods in excess demand. Lemma 7 is used in the proofs of Theorems 1, and 2.

**Lemma 7.** Suppose that Assumptions A1 – A4 hold, and that bidders truthfully report their demand. Then, for each $p \in \mathbb{R}_+^K$

$$E' \in \arg \min_{E \subset K} \sum_{k \in E} (S_k - \sum_{i \in N} \tilde{x}_i^k (p, 1^E))$$

if and only if

$$E' \in \arg \min_{E \subset K} L(p + \delta^{E+}(p)),$$

$$E_+(p) \in \arg \min_{E \subset K} \sum_{k \in E} (S_k - \sum_{i \in N} \tilde{x}_i^k (p, 1^E)),$$  \hspace{1cm} (28)

and for each $\delta \in [0, \delta(p))$

$$E_+(p + \delta^{E+}(p)) = E_+(p).$$

Moreover, if $t$ and $t'$ are two consecutive steps of the extended ascending algorithm such that $t < t'$, then for all $\hat{t} \in [t, t')$

$$E_+(p(\hat{t})) = E_+(p(t)).$$

Similarly,

$$E' \in \arg \min_{E \subset K} \sum_{k \in E} (S_k - \sum_{i \in N} \tilde{y}_i^k (p, 1^E))$$

if and only if

$$E' \in \arg \min_{E \subset K} L(p + \delta^{E+}(p)),$$

$$E_-(p) \in \arg \min_{E \subset K} \sum_{k \in E} (S_k - \sum_{i \in N} \tilde{y}_i^k (p, 1^E)),$$
and for each \( \delta \in [0, \delta(p)) \),
\[
E_-(p - \delta E^E(p)) = E_-(p).
\]

Moreover, if \( t \) and \( t' \) are two consecutive steps of the extended descending algorithm such that \( t < t' \), then for all \( \bar{t} \in [t, t') \)
\[
E_-(p(\bar{t})) = E_-(p(t)).
\]

**Proof.** For each \( p \in \mathbb{R}^K \), by Proposition 3, there exist \( \delta(p) \in \mathbb{R}^+ \) and \( E_+(p) \subset K \) such that \( p + \delta(p)E_+(p) \). As
\[
\delta(p)E_+(p) \cdot x = \delta(p) \sum_{k \in E_+(p)} x^k,
\]
by Proposition 2 and by the definition of \( \tilde{x}_i(p, 1^{E_+(p)}) \), for each \( i \in N \), and for each \( \delta \in (0, \delta(p)] \)
\[
\tilde{x}_i(p, 1^{E_+(p)}) \in Q_i(p + \delta E_+(p)).
\]

Observe that
\[
L(p + \delta E_+(p)) = (p + \delta E_+(p)) \cdot S
\]
\[
+ \sum_{i \in N} (U_i(\tilde{x}_i(p, 1^{E_+(p)}) - (p + \delta E_+(p)) \cdot \tilde{x}_i(p, 1^{E_+(p)})))
\]
\[
= p \cdot S + \sum_{i \in N} (U_i(\tilde{x}_i(p, 1^{E_+(p)}) - p \cdot \tilde{x}_i(p, 1^{E_+(p)})) + \delta E_+(p) \cdot S
\]
\[
- \sum_{i \in N} \delta E_+(p) \cdot \tilde{x}_i(p, 1^{E_+(p)}))
\]
\[
= L(p) + \delta E_+(p) \cdot S - \sum_{i \in N} \delta E_+(p) \cdot \tilde{x}_i(p, 1^{E_+(p)}).
\]

So, minimizing \( L(p + \delta E) \), and maximizing \( L(p) - L(p + \delta E) \) over the set of goods \( E \subset K \) is equivalent to
\[
\min_{E \subset K} \delta \sum_{k \in E} (S_k - \sum_{i \in N} \tilde{x}_i^k(p, 1^E)). \tag{29}
\]

Hence, for each \( \delta, \delta' \in (0, \delta(p)] \), the set of solutions to equation 29 for \( \delta \) and \( \delta' \) are identical.
Thus,
\[ E_+(p) \in \arg \min_{E \subset K} \sum_{k \in E} (S_k - \sum_{i \in N} \tilde{z}_i^k(p, 1^E)), \]

and for each \( \delta \in (0, \delta(p)] \)
\[ E_+(p + \delta E_+(p)) = E_+(p). \]

An analogous derivation can be made for \( E_-(p) \). Observe that at each equilibrium price vector \( p^* \),
\[ E_+(p^*) = E_-(p^*) = \emptyset. \]

Note that Lemma 7 implies that in the extended ascending algorithm and the extended descending algorithm, bidders reporting their demand sets at the initial price vector and at prices at which they add a bundle to their demand sets gives sufficient information to the auctioneer to adjust prices correctly.

Propositions 4 and 5 are generalizations of Propositions 4 and 5 in Ausubel (2006) to economies with bidders who have real-valued utilities. Proposition 4 shows how the price vector at which the extended algorithm terminates relates to the set of Walrasian equilibrium price vectors.

**Proposition 4.** Suppose that Assumptions A1 – A4 hold, and that bidders truthfully report their demand. Starting from any initial price vector \( p(0) \in \mathbb{R}_+^K \), if the extended ascending algorithm ends at \( T \), then \( p(T) \geq p \). Similarly, starting from any initial price \( p(0) \in \mathbb{R}_+^K \), if the extended descending algorithm ends at \( T \), then \( p(T) \leq \bar{p} \).

**Proof of Proposition 4.** It is proven by replacing 1 with \( p(T) \) in the proof of Proposition 4 in Ausubel (2005), p.10.

Proposition 5 shows that if the initial prices are smaller (larger) than the lowest (highest) Walrasian equilibrium price vector, then all the price vectors reached by the extended ascending (descending) algorithm are smaller (larger) than the lowest (highest) Walrasian equilibrium price vector.
Proposition 5. Suppose that Assumptions A1 – A4 hold, and that bidders truthfully report their demand. In the extended ascending algorithm, starting from any initial price vector \( p(0) \in \mathbb{R}^K_+ \), if \( p(t) \leq \bar{p} \), then \( p(t') \leq \bar{p} \) for all \( t' > t \). In the extended descending algorithm, starting from any initial price vector \( p(0) \in \mathbb{R}^K_+ \), if \( p(t) \geq \bar{p} \), then \( p(t') \geq \bar{p} \) for all \( t' > t \).

Proof of Proposition 5. It is proven by modifying the proof of Proposition 5 in Ausubel (2005), pp.10-11 for continuous time price adjustment procedures.

Proof of Theorem 1. Part I. The algorithm converges to a price vector. Consider the extended ascending algorithm. As there are finite number of bidders and the consumption set of each bidder is bounded, there is an upper bound on bidders’ values for each good. In other words, for each good \( k \in K \), there is a price \( p^k_{\text{max}} \in \mathbb{R}_+ \) at and above which no bidder wants good \( k \) regardless of the prices of the rest of the goods. Observe that, by Lemma 7, if the total quantity demanded of a good \( k \in K \) at price vector \( p(t) \in \mathbb{R}^K_+ \) is less than the total quantity of good \( k \) available, then the price of good \( k \) will not be in the set of prices that will be increased at \( t \) in the extended ascending algorithm. Therefore, in the extended ascending algorithm, for each \( k \in K \) and for each \( t \in [0, \infty) \),

\[
p^k(t) \leq p^k_{\text{max}}.
\]

By the Corollary to Proposition 3, for each step \( t' \), there exists a step \( t > t' \) such that there is no other step \( \tilde{t} \) such that \( t' < \tilde{t} < t \), and there exists a rational number \( t_q \in \mathbb{Q} \) such that \( t' < t_q < t \). Therefore, there are at most countably many steps in the extended ascending algorithm. Let \( \{p_s\}_{s \in \sigma} \) be the sequence of all price vectors reached by the extended ascending algorithm at all steps.

Observe that, by construction of the extended ascending algorithm, if \( t' > t'' \), then \( p(t') \geq p(t'') \). Therefore, for each \( s \in \sigma \),

\[
p_s \leq p_{s+1}.
\]

As the sequence \( \{p_s\}_{s \in \sigma} \) is bounded above and monotonically increasing, it converges to a price vector \( p^* \in \mathbb{R}_+ \) such that \( p^* \leq (p^k_{\text{max}})_{k \in K} \).

Part II. The relationship between the demand sets at price vectors of consecutive steps. Note that for each \( s \in \sigma \), \( p_{s+1} \) is the first price vector where a
bidder adds a bundle to his demand set after $p_s$ in the extended ascending algorithm. Since the set of prices that are increased does not change between consecutive steps in the extended ascending algorithm, the same set of prices are increased from $p_s$ to $p_{s+1}$. Therefore, there exist $\delta_s > 0$ and $E_s \subset K$ such that

$$p_{s+1} = p_s + \delta_s^{E_s},$$

which can be rewritten as

$$p_{s+1} = p_0 + \sum_{s'=0}^s \delta_s^{E_{s'}}$$

where $\delta_s > 0$ and $E_s \subset K$ for each $s' \in \mathbb{Z}$ such that $0 \leq s' \leq s$.

For each step $s \in \sigma$, and for each bidder $i_s \in \mathcal{N}$ who adds a bundle to his demand set at price vector $p_{s+1}$, by Lemma 5 for price vectors $p_s$ and $p_{s+1}$, there exist bundles

$$x_{i_s} \in Q_i(p_s) \text{ and } x'_{i_s} \in Q_i(p_{s+1}) \setminus Q_i(p_s)$$

such that

$$\#(x_{i_s} \setminus x'_{i_s}) \leq 1 \text{ and } \#(x'_{i_s} \setminus x_{i_s}) \leq 1,$$ (31)

and there exists a unique $k \in K$ such that $k \in E_s$ and

$$x_{i_s}^k = x_{i_s}^{jk} + 1,$$ (32)

and there exists at most one $k' \in K$ such that $k' \in K \setminus E_s$ and

$$x_{i_s}^{k'} = x_{i_s}^{j k'} - 1.$$

**Part III. The strict monotonicity of excess demand.** For each $s \in \sigma$, let $t_s \in [0, \infty)$ be such that

$$p_s = p(t_s).$$

Now I will show that the total quantity of goods in excess demand strictly decreases at each step, i.e.

$$\sum_{k \in E_s+1} (S_k - \sum_{i \in \mathcal{N}} \bar{x}_i^k(p_{s+1}, 1_{s+1}^E)) > \sum_{k \in E_s} (S_k - \sum_{i \in \mathcal{N}} \bar{x}_i^k(p_s, 1_s^E)).$$ (33)

33
As the prices in $E_s$ are increased from $p_s$ to $p_{s+1}$, by Lemma 3, for each $i \in N$ and for all $t, t' \in (t_s, t_{s+1})$

$$Q_i(p(t)) = Q_i(p(t')),$$

and for each $t \in [t_s, t_{s+1})$

$$Q_i(p_s) \supset Q_i(p(t))$$

and by the definition of step,

$$Q_i(p(t)) \subset Q_i(p_{s+1}).$$

Therefore, for each $t \in (t_s, t_{s+1})$

$$Q_i(p(t)) \subset Q_i(p_s) \cap Q_i(p_{s+1}),$$

and by Lemma 3

$$Q_i(p(t)) = Q_i(p_s) \cap Q_i(p_{s+1}).$$

For each $i \in N$ and for each $t \in (t_s, t_{s+1})$, by equation 34, and by the definition of $\tilde{x}_i(\cdot)$,

$$\sum_{k \in E_{s+1}} \tilde{x}_i^k(p_{s+1}, 1_{s+1}^E) \leq \sum_{k \in E_{s+1}} \tilde{x}_i^k(p(t), 1_{s+1}^E).$$  \hspace{1cm} (35)

Therefore, for each $t \in (t_s, t_{s+1})$

$$\sum_{k \in E_{s+1}} (S_k - \sum_{i \in N} \tilde{x}_i^k(p_{s+1}, 1_{s+1}^E)) \geq \sum_{k \in E_{s+1}} (S_k - \sum_{i \in N} \tilde{x}_i^k(p(t), 1_{s+1}^E)).$$

Since, by Lemma 7, for each $t \in (t_s, t_{s+1})$

$$\sum_{k \in E_{s+1}} (S_k - \sum_{i \in N} \tilde{x}_i^k(p(t), 1_{s+1}^E)) \geq \sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k(p(t), 1_s^E))$$

and

$$\sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k(p(t), 1_s^E)) = \sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k(p_s, 1_s^E)),$$

the following inequality holds

$$\sum_{k \in E_{s+1}} (S_k - \sum_{i \in N} \tilde{x}_i^k(p_{s+1}, 1_{s+1}^E)) \geq \sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k(p_s, 1_s^E)).$$  \hspace{1cm} (36)

Now I will show that the inequality 36 is strict.
Claim. There exists a bidder $i'_s \in N$ such that there does not exist $\tilde{x}_{i'_s} \in Q_{i'_s}(p_s) \cap Q_{i'_s}(p_{s+1})$ satisfying equation 9 at $(p_{s+1}, 1^{E}_{s+1})$.

Proof of the Claim. Suppose, on the contrary, that this is not true. Then, for each $i \in N$ there exists $\tilde{x}_i \in Q_i(p_s) \cap Q_i(p_{s+1})$ satisfying equation 9 at $(p_{s+1}, 1^{E}_{s+1})$. Then,

$$\sum_{k \in E_{s+1}} (S_k - \sum_{i \in N} \tilde{x}_i^k(p_{s+1}, 1^{E}_{s+1})) = \sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k).$$

As

$$\sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k(p_s, 1^E_s)) = \sum_{k \in E_s} (S_k - \sum_{i \in N} \tilde{x}_i^k),$$

by inequality 36, $E_s = E_{s+1}$. But equation 32 implies that for each bidder $i_s$ who adds a bundle at step $t_{s+1}$

$$\sum_{k \in E_s} \tilde{x}_i^k > \sum_{k \in E_s} x^k_{i_s}$$

because

$$\sum_{k \in E_s} x^k_{i_s} > \sum_{k \in E_s} x^k_{i'_s},$$

and

$$\sum_{k \in E_s} \tilde{x}_i^k = \sum_{k \in E_s} x^k_{i_s},$$

where $x'_{i'}$ and $x_{i_s}$ are bundles satisfying equations 30 and inequalities 31. However, $E_s = E_{s+1}$ and inequality 37 imply that $\tilde{x}_{i_s} \in Q_{i_s}(p_s) \cap Q_{i_s}(p_{s+1})$ does not satisfy equation 9 at $(p_{s+1}, 1^{E}_{s+1})$, a contradiction. \hfill \Box

Therefore, there exists a bidder $i'_s \in N$ such that there does not exist $\tilde{x}_{i'_s} \in Q_{i'_s}(p_s) \cap Q_{i'_s}(p_{s+1})$ satisfying equation 9 at $(p_{s+1}, 1^{E}_{s+1})$. So, bidder $i'_s$ adds a bundle to his demand set at $p_{s+1}$, and there exists $x'_{i'_s} \in Q_{i'_s}(p_{s+1})$ satisfying equation 9 at $(p_{s+1}, 1^E_{s+1})$.

Hence,

$$\sum_{k \in E_{s+1}} x^k_{i'_s} < \sum_{k \in E_{s+1}} \tilde{x}^k_{i'_s}$$

for all $\tilde{x}_{i'_s} \in Q_{i'_s}(p_s) \cap Q_{i'_s}(p_{s+1})$. 

35
Inequality 38 implies that inequality 35 is strict for $i'$. So, inequality 36 is strict.

As the left side of inequality 33 is integer and since the extended ascending algorithm terminates whenever it is positive, there are finitely many steps.

Observe that the definition of step implies that if there is no market clearing allocation at a step, then there will not be a market clearing allocation until the next step. Therefore, the extended ascending algorithm terminates at price vector $p_{|\sigma|-1} = p(T) = p^* \in \mathbb{R}_+^K$ for some finite $T$.

The proof for the extended descending algorithm can be done analogously.

\[ \square \]

**Proof of Theorem 2.** The auctioneer asks each bidder $i \in N$ his demand set $x_i(p(0)) \subset X_i$ at $p(0)$. Using these demand sets, the auctioneer determines the set $E_+(p(0)) \subset K$ of goods in excess demand at $p(0)$ (see Lemma 7). Prices of these goods in $E_+(p(0))$ are increased continuously at the same rate while the rest remains the same. As prices are increased, at any time $t \in [0, \infty)$, if there is a bidder who adds a bundle to his demand set, then the price adjustment stops. Each bidder $i$ reports his demand $x_i(p(t)) \subset X_i$ at $p(t)$. The auctioneer determines the set $E_+(p(t)) \subset K$ of goods in excess demand at $p(t)$ (see Lemma 7). Prices of goods in $E_+(p(t))$ are increased continuously at the same rate while the rest remains the same. The extended ascending algorithm reaches some price vector in finite steps (see Theorem 1). By Propositions 4 and 5, the price vector the extended ascending algorithm reaches is $p$, and by Proposition 1, there exists a market clearing allocation $(x^*_i)_{i \in N}$ such that $x^*_i \in x_i(p)$ for each $i \in N$, provided that bidders report their demands truthfully.

\[ \square \]

**Proof of Theorem 4.** It is proven by replacing $1^K$ with $\delta(p)^K$, and modifying the proof of Theorem 2 in Ausubel (2005), p.18, by showing that contradiction is reached for all values of $\delta$.

\[ \square \]

**References**


